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ScienceDirect
Linear Algebra and its Applications 427 (2007) 26–41
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Inherited LU-factorizations of matrices

Marina Arav *, Jean Bevis, Frank J. Hall

Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303, USA

Received 8 December 2006; accepted 17 June 2007
Available online 23 June 2007
Submitted by H. Schneider

Abstract

Various types of LU-factorizations for nonsingular matrices, where L is a lower triangular matrix and U is an upper triangular matrix, are defined and characterized. These types of LU-factorizations are extended to the general $m \times n$ case. The more general conditions are considered in the light of the structures of [C.R. Johnson, D.D. Olesky, P. Van den Driessche, Inherited matrix entries: LU factorizations, SIAM J. Matrix Anal. Appl. 10 (1989) 99–104]. Applications to graphs and adjacency matrices are investigated. Conditions for the product of a lower and an upper triangular matrix to be the zero matrix are also obtained.

AMS classification: 15A23; 15A24; 15A33; 05C50

Keywords: LU-factorization; EZ-factorization; Lower (upper) triangular matrix; Inherited entries; Graphs; Adjacency matrix

1. Introduction

The subject of LU-factorizations of matrices has been an important topic of investigation for a number of years. The applications of LU-factorizations are widespread in analyzing large data sets and are extensively used in areas such as engineering, physics, economics and biology. MathSciNet lists over 300 papers concerning this topic.

* Corresponding author.
E-mail addresses: matmxa@langate.gsu.edu (M. Arav), jhbevis@bellsouth.net (J. Bevis), matfjh@langate.gsu.edu (F.J. Hall).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2007.06.013

While teaching a Linear Algebra course the following observation was made. Some students were computing LU-factorizations incorrectly, namely they were not updating multipliers in Gaussian elimination (they were doing elimination without changing the diagonal entries), but they were getting the right factorization in some problems. This occurrence gave rise to several questions. For what types of matrices will this abbreviated version of elimination work correctly? Are there other abbreviated versions of elimination, or other possibilities for handling the multipliers? This investigation led to LU-factorizations with different groups of inherited or *nearly* inherited entries.

We consider LU-factorizations of a matrix A, where the entries of A are inherited in L or U. Our characterizations involve the submatrices of a specific matrix. On the other hand, the authors in [4] give graph theoretic conditions for inherited LU-factorizations for matrices with a particular zero/nonzero pattern. Their conditions are of generic type, while we obtain more general conditions on the submatrices of the matrices. The conditions we obtain are quantitative in nature, whereas those in [4] are qualitative.

In Section 2, we define and characterize various types of LU-factorizations for nonsingular matrices, where L is lower triangular and U is upper triangular. These types of LU-factorizations are extended to the general $m \times n$ case in Section 3. Conditions for the product of a lower and an upper triangular matrix to be the zero matrix are obtained in Section 4. These results are motivated by the fact that such a product being zero enters into necessary and sufficient conditions for the existence of some of the factorizations examined in Section 2. In Section 5, we note how the more general conditions given in this paper are satisfied in the structures described in [4]. In Section 6, we give applications of our results to graphs and adjacency matrices, which are considered in [1,2]. The notion of an EZ-graph of order r is defined and used in characterizing when the adjacency matrix of a graph (partitioned into 2×2 submatrices) has an "EZ-factorization".

2. LU-factorizations of nonsingular matrices

In this section and in Section 3, we work in the general setting of matrices over a ring $\Re$ with unity. We will eventually apply some of the results to LU-factorizations of adjacency matrices of graphs in Section 6. In this section we assume that A is $n \times n$ nonsingular over the ring $\Re$, $a_{11}, a_{22}, \ldots, a_{nn}$ are invertible elements of $\Re$, and write

$$A = B + D + C,$$

where B is strictly lower triangular, D is diagonal, and C is strictly upper triangular.

We introduce the following factorizations:

F_1: $A = L(D' + C)$ for some unit diagonal, lower triangular L and some diagonal matrix D',
F_1': $A = (B + D')U$ for some unit diagonal, upper triangular U and some diagonal matrix D',
F_2: $A = L(D + C)$ for some unit diagonal, lower triangular L,
F_2': $A = (B + D)U$ for some unit diagonal, upper triangular U,
F_3: $A = (I + BD^{-1})(D + C)$,
F_3': $A = (B + D)(I + D^{-1}C)$,
F_4: $A = (I + B)(D + C)$,
F_4': $A = (B + D)(I + C)$.

Remarks

1. Note the subtle difference between the factorizations F_1 and F_2 and the factorizations F_1' and F_2'. The diagonal matrix D comes from the matrix A, while the diagonal matrix D' is arbitrary.
2. The factorizations F_3, F_3', F_4 and F_4' can be simply written down directly from the decomposition $A = B + D + C$.
3. We note that in each of the F_1, F_1', F_2, F_2', F_3, F_3', F_4 and F_4' factorizations of the matrix A, the lower (or strictly lower) and/or upper (or strictly upper) triangular part of the matrix A is inherited.
4. It can be shown that when A has any of the F_2, F_3, or F_4 type LU-factorizations each of the L and U factors is unique. This holds since in all three cases L is lower triangular with unit diagonal and U is upper triangular with invertible diagonal elements. So, for example, if A has an F_2 factorization and an F_3 factorization, L must be $I + BD^{-1}$. Similarly for F_2', F_3', and F_4'. The uniqueness of the L and U factors is also true in the more general $m \times n$ case addressed in Section 3.
5. We can however show that F_2 is actually more general than F_3 or F_4, and also that A may have an F_2 factorization without having an F_2' factorization.

Consider

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = B + D + C.$$

Now,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = L(D + C)$$

is an F_2 type factorization. But, we do not have $A = (B + D)U$ for any unit diagonal, upper triangular matrix U. For if so, we would have

$$L = B + D \quad \text{and} \quad D + C = U.$$

But, $L \neq B + D$, and so we have a contradiction. So, we do *not* have an F_2' factorization.

Also, we do *not* have an F_3 (or F_4) factorization. Thus, F_2 holds, but F_2', F_3, and F_4 do not hold! (We abbreviate "A has an F_i factorization" by just saying that "F_i holds".)

On the other hand, we have the following implications.

Proposition 2.1. *Let A be an $n \times n$ nonsingular matrix with invertible diagonal entries, and write $A = B + D + C$, where B is strictly lower triangular, D is diagonal, and C is strictly upper triangular. Then, the following are equivalent:*

(i) *A has an F_3 factorization.*
(ii) *A has an F_3' factorization.*
(iii) *A has an F_2 factorization and an F_2' factorization.*

Proof. (i) $\Leftrightarrow$ (ii). F_3 and F_3' are equivalent, based on the following:

$$(I + BD^{-1})(D + C) = (I + BD^{-1})D(I + D^{-1}C) = (B + D)(I + D^{-1}C).$$

(ii) $\Rightarrow$ (iii). Suppose F_3 (and therefore F_3') holds. Since $(I + BD^{-1})$ is unit diagonal, lower triangular and $(I + D^{-1}C)$ is unit diagonal, upper triangular, we get F_2 and F_2', correspondingly.

(iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii). Suppose F_2 and F_2' hold. Then

$$A = L(D + C) = LD(I + D^{-1}C),$$
$$A = (B + D)U = (I + BD^{-1})DU.$$

By "uniqueness of LDU factorizations" (see for example [6, Theorem 1.7.28, p. 84]), we get $L = I + BD^{-1}$ and $U = I + D^{-1}C$, so that F_3 (and F_3') holds. □

A factorization of the form F_3 is called an EZ-factorization, and may be described as *U is inherited and L is nearly inherited*. Hence, when A is an $n \times n$ nonsingular matrix, A has an LU-factorization with "U inherited" (F_2) and an LU-factorization with "L inherited" (F_2') if and only if A has an EZ-factorization (F_3).

We observe that when A has an EZ-factorization (F_3), *easy* rules produce both factors L and U. Such matrices should be of interest in sparse matrix theory since there is no fill-in during Gaussian elimination. An EZ-factorization does depend on the underlying ring $\Re$. For example,

$$A = \begin{bmatrix} 2 & 0 & 12 & 6 \\ 0 & 3 & 12 & 6 \\ 2 & -3 & 1 & 0 \\ -6 & 9 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -3 & 3 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 12 & 6 \\ 0 & 3 & 12 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an EZ-factorization over the real number field, but A does not have an EZ-factorization over the ring of integers since D^{-1} is not integral. We also remark that if A is an $n \times n$ integer matrix that has an EZ-factorization, then A is integrally nested, see [3,5].

Theorem 2.2. *Let $A = B + D + C$ be a decomposition of a nonsingular matrix with invertible diagonal entries where B is strictly lower triangular, D is diagonal, and C is strictly upper triangular.*

(i) *A has an F_1 factorization if and only if there is a strictly lower triangular matrix X the same size as A such that $(I + X)^{-1}(B + D - XC)$ is diagonal. For such a matrix the product XC must be lower triangular.*
(ii) *A has an F_2 factorization $A = L(D + C)$ if and only if $B(D + C)^{-1}$ is strictly lower triangular.*
(iii) *A has an F_3 factorization $A = (I + BD^{-1})(D + C)$ if and only if $BD^{-1}C = O$.*
(iv) *A has an F_4 factorization $A = (I + B)(D + C)$ if and only if $B(I - D - C) = O$.*

Proof. Suppose there is a matrix X satisfying the condition of part (i) and set $Y = (I + X)^{-1}(B + D - XC)$. Then $Y + XY = B + D - XC$ and $Y + C + XY + XC = B + D + C$. But this last equality may be written as $(I + X)(Y + C) = A$, which is an F_1 factorization of A. The computation may be reversed to obtain the necessity of this condition. When these conditions hold $XC = D - Y + B - XY$, which is lower triangular.

In (ii) L is a unit lower triangular matrix, so $L - I$ is strictly lower triangular. Now $A = L(D + C)$ is equivalent to $B + D + C = (L - I)(D + C) + (D + C)$, which in turn is equivalent to $B(D+C)^{-1} = L - I$. Now, L can be chosen to satisfy this last equality if and only if $B(D+C)^{-1}$ is strictly lower triangular, so (ii) is obtained.

Similarly, $B + D + C = (I + BD^{-1})(D + C) = D + C + B + BD^{-1}C$ is equivalent to $BD^{-1}C = O$ for (iii), and $B + D + C = (I + B)(D + C) = D + C + BD + BC$ is equivalent to $B(I - D - C) = O$ for (iv). $\square$

Note that all of these characterizations involve a product of a lower triangular matrix and an upper triangular matrix (in some cases they are strictly triangular), and require that the product be triangular, strictly triangular, diagonal, or zero. Even the condition in (i) for the existence of the matrix X can be written as: $(D - Y + B)(Y + C)^{-1}$ is strictly lower triangular for some diagonal matrix Y. This gives rise to several interesting matrix equations. In Section 4 we start an investigation on the construction of solutions to a few of these equations. But first we consider the general $m \times n$ case, and present results extending F_2, F_2', F_3, F_3', F_4, and F_4'.

3. The general $m \times n$ case

We first generalize the F_2 factorizations.

Definition 3.1. Let A be an $m \times n$ matrix and write

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where A_1 is $r \times r$ and $r \leqslant \min(m, n)$.

Further, write $A_1 = B_1 + D_1 + C_1$, where B_1 is strictly lower triangular, D_1 is diagonal, and C_1 is strictly upper triangular. Then, A has an order r left unit LU-factorization with U inherited if and only if there exists an $m \times r$ unit lower triangular matrix L such that

$$A = L\begin{bmatrix} D_1 + C_1 & A_2 \end{bmatrix}.$$

Remarks

1. In the above definition, unit matrix L means that all $l_{ii} = 1$.
2. An $m \times r$ ($r \times n$) lower (upper) triangular matrix is sometimes referred to as a trapezoidal matrix. However, as in previous papers such as [1,2], we will use the term lower (upper) triangular.

Theorem 3.2. *Let A be an $m \times n$ matrix and write $A = \begin{bmatrix} B_1 + D_1 + C_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ as in Definition 3.1. Suppose that $a_{11}, a_{22}, \ldots, a_{rr}$ are invertible elements of the ring $\Re$. Then, A has an order r left unit LU-factorization with U inherited if and only if*

(i) $B_1(D_1 + C_1)^{-1}$ *is strictly lower triangular*,
(ii) $B_1(D_1 + C_1)^{-1}A_2 = 0$,
(iii) *there exists an $(m - r) \times r$ matrix E such that $A_3 = EA_1$ and $A_4 = EA_2$.*

Proof. ($\Rightarrow$) Suppose A has such a factorization. Then there exists an $m \times r$ unit lower triangular matrix L such that

$$A = L\begin{bmatrix} D_1 + C_1 & A_2 \end{bmatrix}.$$

Write

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix},$$

where L_1 is $r \times r$. Then, $A_1 = B_1 + D_1 + C_1 = L_1(D_1 + C_1)$, and since $a_{11}, \ldots, a_{rr}$ are invertible, $D_1 + C_1$ is an invertible upper triangular matrix.

Hence, $L_1 = I + B_1(D_1 + C_1)^{-1}$, so that $B_1(D_1 + C_1)^{-1}$ must be strictly lower triangular. Also, $A_2 = L_1 A_2 = [I + B_1(D_1 + C_1)^{-1}]A_2$ implies that $B_1(D_1 + C_1)^{-1}A_2 = 0$.

Next, define $E = L_2 L_1^{-1}$. Then

$$EA_1 = L_2 L_1^{-1} A_1 = L_2(D_1 + C_1) = A_3$$

and

$$EA_2 = EL_1 A_2 = L_2 A_2 = A_4.$$

($\Leftarrow$) We assume (i)–(iii) hold. Define $L_1 = I + B_1(D_1 + C_1)^{-1}$. Then L_1 is $r \times r$ unit lower triangular and $L_1 \begin{bmatrix} D_1 + C_1 & A_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$ from (i)–(ii).

Using the matrix E given in (iii), define $L_2 = EL_1$. Then

$$A_3 = EA_1 = EL_1(D_1 + C_1) = L_2(D_1 + C_1)$$

and

$$A_4 = EA_2 = EL_1 A_2 = L_2 A_2.$$

Thus, with $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$, L is $m \times r$ unit lower triangular and $A = L \begin{bmatrix} D_1 + C_1 & A_2 \end{bmatrix}$. $\square$

Observing the proof of Theorem 3.2, L_1 is determined as $L_1 = I + B_1(D_1 + C_1)^{-1}$. Now, $A_3 = L_2(D_1 + C_1)$, so that $L_2 = A_3(D_1 + C_1)^{-1}$ is determined. Hence, L is determined. Of course, $U = \begin{bmatrix} D_1 + C_1 & A_2 \end{bmatrix}$. Thus, when A has an order r left unit LU-factorization with U inherited, the L and U factors are uniquely determined. The uniqueness of the L and U factors is also easily seen for the other types of LU-factorizations in this section.

We have a similar definition for an order r right unit LU-factorization with L inherited (thus generalizing F_2').

Definition 3.3. Let A be an $m \times n$ matrix and write

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad \text{where } A_1 \text{ is } r \times r \text{ and } r \leqslant \min(m, n).$$

Further, write $A_1 = B_1 + D_1 + C_1$, where B_1 is strictly lower triangular, D_1 is diagonal, and C_1 is strictly upper triangular. Then, A has an order r right unit LU-factorization with L inherited if and only if there exists an $r \times n$ unit upper triangular matrix U such that

$$A = \begin{bmatrix} B_1 + D_1 \\ A_3 \end{bmatrix} U.$$

Similar to the proof of Theorem 3.2, the following result can be obtained.

Theorem 3.4. *Let A be an $m \times n$ matrix and write $A = \begin{bmatrix} B_1 + D_1 + C_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ as in Definition 3.1. Suppose that $a_{11}, a_{22}, \ldots, a_{rr}$ are invertible elements of the ring $\Re$. Then, A has an order r right unit LU-factorization with L inherited if and only if*

(i) $(D_1 + B_1)^{-1}C_1$ *is strictly upper triangular*,
(ii) $A_3(D_1 + B_1)^{-1}C_1 = 0$,
(iii) *there exists an* $r \times (n-r)$ *matrix* F *such that* $A_2 = A_1F$ *and* $A_4 = A_3F$.

We next consider EZ-factorizations, and extend the F_3 factorizations.

Definition 3.5. Let A be an $m \times n$ matrix and $a_{11}, a_{22}, \dots, a_{rr}$ be invertible elements of the ring $\Re$, where $r \leqslant \min(m, n)$. Then, A has an order r EZ-factorization with U inherited if and only if $A = LU$ for the $m \times r$ unit lower triangular matrix L with $l_{ij} = a_{ij}a_{jj}^{-1}$ for $i \geqslant j$ and the $r \times n$ upper triangular matrix U with $u_{ij} = a_{ij}$ for $i \leqslant j$.

We again partition A as

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where $A_1 = B_1 + D_1 + C_1$ is $r \times r$, B_1 is strictly lower triangular, D_1 is diagonal, and C_1 is strictly upper triangular. Assume $a_{11}, a_{22}, \dots, a_{rr}$ are invertible and set

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} I + B_1D_1^{-1} \\ A_3D_1^{-1} \end{bmatrix}, \quad U = \begin{bmatrix} U_1 & U_2 \end{bmatrix} = [D_1 + C_1 \quad A_2],$$

so that L is $m \times r$ unit lower triangular with $l_{ij} = a_{ij}a_{jj}^{-1}$ for $i > j$ and U is $r \times n$ upper triangular with $u_{ij} = a_{ij}$ for $i \leqslant j$. Then, A has an order r EZ-factorization with U inherited if and only if $A = LU$.

Now,

$$LU = \begin{bmatrix} B_1 + D_1 + C_1 + B_1D_1^{-1}C_1 & A_2 + B_1D_1^{-1}A_2 \\ A_3 + A_3D_1^{-1}C_1 & A_3D_1^{-1}A_2 \end{bmatrix}.$$

Hence, A has an order r EZ-factorization with U inherited if and only if $B_1D_1^{-1}C_1 = 0$, $B_1D_1^{-1}A_2 = 0$, $A_3D_1^{-1}C_1 = 0$, and $A_3D_1^{-1}A_2 = A_4$.

With A, L, and U as above, further assume that there exists an $(m-r) \times r$ matrix E such that $A_3 = EA_1$ and $A_4 = EA_2$. (If A has an EZ-factorization, Theorem 3.2 guarantees that such a matrix E exists.) Suppose that $B_1D_1^{-1}C_1 = 0$, $B_1D_1^{-1}A_2 = 0$, and $A_3D_1^{-1}C_1 = 0$.

Then, $A_1 = L_1U_1$, $A_2 = L_1U_2$, and $A_3 = L_2U_1$. Also, $EA_1 = EL_1U_1$ and $EA_1 = A_3 = L_2U_1$.

Since U_1 is an invertible upper triangular matrix, we have $EL_1 = L_2$ and so

$$L_2U_2 = EL_1U_2 = EA_2 = A_4.$$

We arrive at the following result.

Theorem 3.6. *Let* A *be an* $m \times n$ *matrix and write* $A = \begin{bmatrix} B_1 + D_1 + C_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ *as in Definition* 3.1. *Suppose that* $a_{11}, a_{22}, \dots, a_{rr}$ *are invertible elements of the ring* $\Re$. *Then, the following are equivalent*:

(i) *A has an order* r EZ*-factorization with* U *inherited*.
(ii) $B_1D_1^{-1}C_1 = 0$, $B_1D_1^{-1}A_2 = 0$, $A_3D_1^{-1}C_1 = 0$, *and* $A_3D_1^{-1}A_2 = A_4$.

(iii) $B_1D_1^{-1}C_1 = 0$, $B_1D_1^{-1}A_2 = 0$, $A_3D_1^{-1}C_1 = 0$, *and there exists an* $(m-r) \times r$ *matrix* E *such that* $A_3 = EA_1$ *and* $A_4 = EA_2$.

Next observe that

A has an EZ-factorization with U inherited
if and only if
$A = LU$ for the $m \times r$ unit lower triangular matrix L with $l_{ij} = a_{ij}a_{jj}^{-1}$ for $i \geqslant j$ and the $r \times n$ upper triangular matrix U with $u_{ij} = a_{ij}$ for $i \leqslant j$
if and only if (by the associativity of matrix multiplication)
$A = LU$ for the $m \times r$ lower triangular matrix L with $l_{ij} = a_{ij}$ for $i \geqslant j$ and the $r \times n$ unit upper triangular matrix U with $u_{ij} = a_{ii}^{-1}a_{ij}$ for $i \leqslant j$
if and only if
A has an "EZ-factorization with L inherited" (the generalization of F_3').

Hence, A has an EZ-factorization with U inherited if and only if A has an EZ-factorization with L inherited. Thus, F_3 holds if and only if F_3' holds in the general $m \times n$ case.

We saw in Section 2 that a square matrix A has an F_2 type factorization and an F_2' type factorization if and only if A has an F_3 type factorization. For $m \times n$ matrices, a natural question is the following. Is it true that A has an order r left unit LU-factorization with U inherited and an order r right unit LU-factorization with L inherited if and only if A has an order r EZ-factorization with U inherited? This is still an open question.

Finally, we generalize the F_4 factorizations.

Definition 3.7. Let A be an $m \times n$ matrix and $a_{11}, a_{22}, \ldots, a_{rr}$ be invertible elements of the ring $\Re$, where $r \leqslant \min(m, n)$. Then, A has an order r G.E. factorization with U inherited if and only if $A = LU$ for the $m \times r$ unit lower triangular matrix L with $l_{ij} = a_{ij}$ for $i > j$ and the $r \times n$ upper triangular matrix U with $u_{ij} = a_{ij}$ for $i \leqslant j$.

"G.E. factorization" refers to the fact that in this case $A = LU$ is set for Gaussian elimination. The proof of the following characterization is similar to the proof of Theorem 3.6 and is omitted. In this case,

$$L = \begin{bmatrix} I + B_1 \\ A_3 \end{bmatrix}$$

and U remains as $[D_1 + C_1 \quad A_2]$.

Theorem 3.8. *Let A be an $m \times n$ matrix and write* $A = \begin{bmatrix} B_1 + D_1 + C_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ *as in Definition 3.1. Suppose* $a_{11}, a_{22}, \ldots, a_{rr}$ *are invertible elements of* $\Re$. *Then, the following are equivalent*:

(i) *A has an order r G.E. factorization with U inherited.*
(ii) $B_1[I - D_1 - C_1] = 0$, $A_3[I - D_1 - C_1] = 0$, $B_1A_2 = 0$, *and* $A_3A_2 = A_4$.
(iii) $B_1[I - D_1 - C_1] = 0$, $A_3[I - D_1 - C_1] = 0$, $B_1A_2 = 0$, *and there exists an* $(m-r) \times r$ *matrix* E *such that* $A_3 = EA_1$ *and* $A_4 = EA_2$.

There is of course a similar characterization for matrices that have an "order r G.E. factorization with L inherited" (generalizing F_4').

4. Solutions of the matrix equation $BC = 0$

For simplicity, in this section and in Section 5, we assume that our matrices are square and have real entries. In this section we will consider the equation $BC = 0$ where B and C denote $n \times n$ lower (respectively upper) triangular matrices. Such a product being zero enters into necessary and sufficient conditions for the existence of the F_3 and F_4 factorizations examined in Section 2. Now $BC = 0$ if and only if every column of B^{T} is orthogonal to every column of C. But the only possible nonzero entries in the first columns of B^{T} and C are b_{11} and c_{11}. Hence if $BC = 0$, then at least one of b_{11} or c_{11} must be 0. This leads us to

Observation 1. If $BC = 0$ and $b_{11} \neq 0$, then the entire first row of C consists of zeros. Similarly, if $BC = 0$ and $c_{11} \neq 0$, then the entire first column of B consists of zeros.

If this pattern is followed in the remaining parts of B and C, one can obtain a pair of triangular matrices where for $k = 1, 2, \ldots, n$ either column k of B or row k of C consists entirely of zeros. In this case $BC = 0$, and the matrices B and C are said to exhibit a saw tooth pattern. Theorem 5.2 shows that this condition is equivalent to $BC = O$, where the equality holds generically. That is to say, the equality holds for any matrices with the same zero/nonzero patterns as B and C, respectively.

If both B and C contain leading principal submatrices consisting entirely of zeros, then other matrix structures are available. Partition B and C as

$$B = \begin{bmatrix} B_1 & 0 \\ B_2 & B_3 \end{bmatrix}, \qquad C = \begin{bmatrix} C_1 & C_2 \\ 0 & C_3 \end{bmatrix}. \tag{I}$$

Theorem 4.1. *If $B_1 = C_1 = 0$ are $k \times k$, then B, C are lower (respectively upper) triangular matrices with $BC = 0$ if and only if B_3 and C_3 are lower (respectively upper) triangular matrices and B_2C_2 is a factorization of $-B_3C_3$. Furthermore, for fixed submatrices B_3 and C_3, there exist submatrices B_2 and C_2 such that $BC = 0$ if and only if $\operatorname{rank}(B_3C_3) \leqslant k$.*

Proof. If $BC = 0$ then $B_2C_2 = -B_3C_3$, which necessarily has rank less than or equal to k. Conversely, if $\operatorname{rank}(B_3C_3) \leqslant k$, then $\operatorname{rank}(-B_3C_3) \leqslant k$. Hence, $-B_3C_3$ has a factorization into a product of an $(n-k) \times k$ matrix times a $k \times (n-k)$ matrix. In order to obtain such a product, we can first factor $-B_3C_3$ into a full-rank factorization GH, where G is $(n-k) \times r$ and H is $r \times (n-k)$. Then, append $k - r$ zero columns (rows) to G (H). $\square$

Observation 2. If $k \geqslant n/2$ in Theorem 4.1, then for any $(n-k) \times (n-k)$ lower (upper) triangular matrices B_3 and C_3, respectively, there exist submatrices B_2 and C_2 such that $BC = 0$.

If B or C is 0 then $BC = 0$ trivially, so consider the case where they are both nonzero. Let t be one more than the size of the largest zero matrix that occurs as a leading principal submatrix in each of B and C. Take B_1 and C_1 in (I) to be $t \times t$, so that only the last row of B_1 or the last column of C_1 is nonzero. As t increases from 1 as in Observation 1 to at least $n/2 + 1$ in Observation 2, there are two trends that may be observed. One is that the conditions on B_3 and C_3 decrease from B_3 and C_3 are lower (respectively upper) triangular matrices with $B_3C_3 = 0$, to B_3 and C_3 can be arbitrary lower (respectively upper) triangular matrices. The other trend is that the relationship between B_2, C_2 and B_3, C_3 increases from total independence to highly dependent. The next theorem sheds some light on these trends by providing an intermediary step in this progression based on a pattern determined by a 2-vector $(p, q)^{\mathrm{T}}$ and its orthogonal complement $(-q, p)^{\mathrm{T}}$.

Theorem 4.2. *Let B and C be $n \times n$ lower (respectively upper) triangular matrices. Let t be one more than the size of the largest zero matrix that occurs as a leading principal submatrix in each of B and C. Partition B and C as in* (I) *where B_1 and C_1 are $t \times t$ in parts* (i) *and* (ii), *but in part* (iii) *they are $(t-1) \times (t-1)$.*

(i) *If $t = 1$, then $BC = 0$ if and only if $B_3C_3 = 0$ and either ($b_{11} = 0$ and $B_2 = 0$) or ($c_{11} = 0$ and $C_2 = 0$).*
(ii) *If $t = 2$, then $BC = 0$ if and only if there is a nonzero vector $\upsilon = (p, q)^{\mathrm{T}}$ such that the last row of B_1 is a multiple of υ^{T}, the last column of C_1 is a multiple of $\upsilon^{\perp} = (-q, p)^{\mathrm{T}}$, and at least one of* (ii$_\mathrm{a}$), (ii$_\mathrm{b}$), *or* (ii$_\mathrm{c}$) *is true.*
  (ii$_\mathrm{a}$) *$B_3C_3 = 0$, and the columns of B_2 and rows of C_2 exhibit a saw tooth pattern.*
  (ii$_\mathrm{b}$) *$B_3C_3 = 0$, and there exist column vectors β, γ of length $n-2$ such that $B_2 = [p\beta \quad q\beta]$ and $C_2 = \begin{bmatrix} -q\gamma^{\mathrm{T}} \\ p\gamma^{\mathrm{T}} \end{bmatrix}$.*
  (ii$_\mathrm{c}$) *$B_3C_3 = -B_2C_2$, and $B_1 = 0$, $B_2 = [p\beta \quad q\beta]$ for some column vector β of length $n-2$ (by necessity $B_3C_3 = -B_2C_2$ has rank at most 1,) or $C_1 = 0$, $C_2 = \begin{bmatrix} -q\gamma^{\mathrm{T}} \\ p\gamma^{\mathrm{T}} \end{bmatrix}$ for some column vector γ of length $n-2$.*
(iii) *If $t \geqslant \frac{n}{2} + 1$, then $BC = 0$ if and only if B_3, C_3 are any lower (respectively upper) triangular matrices, and $B_2C_2 = -B_3C_3$.*

Proof. The cases where $t = 1$ and $t \geqslant \frac{n}{2} + 1$ are essentially covered in Observations 1 and 2. If $t = 2$, then $B_1C_1 = 0$ if and only if the last row of B_1 and the last column of C_1 are orthogonal vectors in $\mathbb{R}^2$. That is, if and only if there is a nonzero vector $\upsilon = (p, q)^{\mathrm{T}}$ such that the last row of B_1 is a multiple of υ^{T}, the last column of C_1 is a multiple of $\upsilon^{\perp} = (-q, p)^{\mathrm{T}}$. Also, if $t = 2$, at least one of B_1 or C_1 is nonzero. If B_1 and B_2, or C_1 and C_2, are both zero matrices, then the first two rows and columns of B and C exhibit a saw tooth pattern. Further, $BC = 0$ in this case if and only if $B_3C_3 = 0$. This gives us the result (ii$_\mathrm{a}$). If B_1 is nonzero, then $B_1C_2 = 0$ if and only if each column of C_2 is orthogonal to υ, equivalently, for $k = 1, 2, \dots, n-2$, column k of C_2 is $\gamma_k\upsilon^{\perp}$ for some scalar γ_k. Let $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_{n-2})^{\mathrm{T}}$. Now the first entry in the kth column of C_2 is $-q\gamma_k$ and the second entry in that column is $p\gamma_k$, so C_2 has the form $\begin{bmatrix} -q\gamma^{\mathrm{T}} \\ p\gamma^{\mathrm{T}} \end{bmatrix}$ which is used in (ii$_\mathrm{b}$) and (ii$_\mathrm{c}$). Similarly, if C_1 is nonzero, then $B_2C_1 = 0$ if and only if B_2 has the form $[p\beta \quad q\beta]$ which is used in (ii$_\mathrm{b}$) and (ii$_\mathrm{c}$). Now if both B_1 and C_1 are nonzero, then B_2 and C_2 have the given forms and as a result $B_2C_2 = 0$. Further, $BC = 0$ in this case if and only if $B_3C_3 = 0$. This gives the result (ii$_\mathrm{b}$). Next consider the case, where $C_1 = 0$, but B_1 and C_2 are both nonzero. As before $B_1C_2 = 0$ if and only if C_2 has the form given in (ii$_\mathrm{c}$). Further, $BC = 0$ if and only if $B_3C_3 = -B_2C_2$, which is rank one in this case, since each row is a multiple of γ^{T}. The case, where $B_1 = 0$, but C_1 and B_2 are both nonzero is similar to this last one, and together they give the result (ii$_\mathrm{c}$). □

This result suggests several methods of constructing larger triangular matrices with a zero product from smaller ones by adding initial rows and columns. It is also possible to do this by adding final rows and columns. Suppose that we start with lower (respectively upper) triangular matrices B', C' such that $B'C' = 0$. To B' add a final column of zeros and then a final row consisting of (β^{T}, p), where p is a scalar to be determined and $\beta \in N(C'^{\mathrm{T}})$, the null space of

C'^{T}. To C' add a final row of zeros and then a final column consisting of $\begin{bmatrix}\gamma\\ q\end{bmatrix}$, where q is a scalar to be determined and $\gamma \in N(B')$. Finally, choose p and q such that $pq = -\beta^{\mathrm{T}}\gamma$. Now B, C are lower (respectively upper) triangular matrices and

$$BC = \begin{bmatrix} B' & 0 \\ \beta^{\mathrm{T}} & p \end{bmatrix}\begin{bmatrix} C' & \gamma \\ 0 & q \end{bmatrix} = \begin{bmatrix} B'C' & B'\gamma \\ \beta^{\mathrm{T}}C' & \beta^{\mathrm{T}}\gamma + pq \end{bmatrix} = 0.$$

The results of this section can also be adapted to strictly triangular matrices. Let B' and B be matrices, where B can be obtained from B' by adding an initial row and final column of zeros. Now B' is lower triangular if and only if B is strictly lower triangular. Similarly, let C' and C be matrices, where C can be obtained from C' by adding a final row and initial column of zeros. Now C' is upper triangular if and only if C is strictly upper triangular. Further more $B'C' = 0$ if and only if $BC = 0$.

5. Comparison with [4]

In [4] the authors characterize matrices $A = LU$ with LU-factorizations where the entries of L or U are inherited from A and the equality $A = LU$ holds generically. That is, if A has an LU-factorization with a certain collection of entries inherited, then the same is true of all matrices with the same zero/nonzero structure as A. This means that only certain matrices having an LU-factorization with inherited entries are considered, and these satisfy such a property because of the structure of their underlying digraph, not the relative sizes of their nonzero entries. The underlying digraph of an $n \times n$ matrix A, denoted by $d(A)$, is a digraph on the set of vertices $1, 2, \ldots, n$ with an edge from i to j provided that $a_{ij} \neq 0$. Further, they only consider square matrices, where the proper leading principal minors are nonzero so that the LU-factorization is unique. It is interesting to note how the more general conditions given in this paper are satisfied in the structures described in.

For simplicity we restrict our attention to $n \times n$ nonsingular matrices with nonzero leading principal minors. In [4, Corollary 3.4] the authors characterize such matrices $A = B + D + C$ having an F_2 factorization or (left) unit LU-factorization $A = LU$ with $U = D + C$ where the equality $U = D + C$ holds generically. Such matrices are characterized as being (i, j)-*lower restricted* for all pairs (i, j) with $1 \leqslant i \leqslant j \leqslant n$. This means that in the digraph $d(A)$, there is no path of length two or more from i to j with all intermediate vertices less than i for all $i \leqslant j$. Equivalently, all products $a_{i,k_1}a_{k_1,k_2}\ldots a_{k_l,j}$ are zero for all $i \leqslant j$ whenever $k_1, k_2, \ldots, k_l < i$. Furthermore, from [4], this (i, j)-lower restricted property of A implies that all $a_{ii} \neq 0$ if A is nonsingular.

In Theorem 2.2(ii) above we found the condition $B(D + C)^{-1}$ is strictly lower triangular, is equivalent to $U = D + C$ with the usual meaning for the equality. To compare these results we must expand $B(D + C)^{-1}$ as a finite power series $B(D + C)^{-1} = \sum_{k=0}^{n-1}(-1)^k BD^{-1}(CD^{-1})^k$. The entries on or above the diagonal of the matrix are then sums or differences of terms of the form

$$b_{i,k_1}d_{k_1,k_1}^{-1}c_{k_1,k_2}d_{k_2,k_2}^{-1}\ldots c_{k_l,j}d_{j,j}^{-1} = a_{i,k_1}a_{k_1,k_2}\ldots a_{k_l,j}d_{k_1,k_1}^{-1}d_{k_2,k_2}^{-1}\ldots d_{j,j}^{-1}.$$

Since B is strictly lower triangular, we need only consider values of i and k_1 with $k_1 < i$.

Similarly, since C is strictly upper triangular, we can restrict the other indices to ranges with $k_1 < k_2 < k_3 < \cdots < k_l < j$.

Since $k_1 < i$, for $i \leqslant j$, there must be a first index k' or j in the sequence $k_1, k_2, \ldots, k_l, j$ for which $i \leqslant k'$, or $k_1 < k_2 < \cdots < k_l < i \leqslant j$. Now if $U = D + C$ holds generically then A is (i, k')-lower restricted and the displayed product is 0.

Thus $U = D + C$ holds generically if and only if each $BD^{-1}(CD^{-1})^k$, or equivalently each BC^k, $k = 1, 2, \ldots, n-1$ is generically equal to a strictly lower triangular matrix.

We thus obtain:

Theorem 5.1. *Let $A = B + D + C$ be an $n \times n$ nonsingular matrix with nonzero leading principal minors so that A has a unique left unit LU-factorization $A = LU$. The following are equivalent*:

(i) *$U = D + C$ with the equality holding generically.*
(ii) *$B(D + C)^{-1}$ is strictly lower triangular holds generically.*
(iii) *A is (i, j)-lower restricted for all pairs (i, j) with $1 \leqslant i \leqslant j \leqslant n$.*
(iv) *BC^k is generically equal to a strictly lower triangular matrix, for $1 \leqslant k \leqslant n - 1$.*

Proof. $U = D + C$ holds generically is equivalent to the equality holding for any matrix with the same zero pattern which, by Theorem 2.2(ii), is equivalent to $B(D + C)^{-1}$ is strictly lower triangular for any matrix with the same zero pattern. Thus (i) and (ii) are equivalent. The equivalence of (i) and (iii) is given by [4, Corollary 3.4], and the equivalence of (i) and (iv) is given in the discussion above. □

Next consider the F_3-factorization. In [4, Theorem 4.1] the authors show that a square matrix with nonzero leading principal minors has such a factorization generically if and only if the matrix exhibits a saw tooth pattern. Using this result, Theorem 2.2(iii), and an argument similar to the one above, one obtains:

Theorem 5.2. *Let $A = B + D + C$ be a $n \times n$ nonsingular matrix with nonzero leading principal minors so that A has a unique left unit LU-factorization $A = LU$. The following are equivalent*:

(v) *$U = D + C$ and $L = I + BD^{-1}$ with the equalities holding generically.*
(vi) *$BD^{-1}C = 0$ holds generically.*
(vii) *A exhibits a saw tooth pattern.*

6. Applications to graphs and adjacency matrices

Let $G = (V, E)$ be a graph with vertex set V and edge set E, where there are no loops or multiple edges. Each edge is incident with two vertices. If edge e is incident with vertices u and v, then e can be represented by the unordered pair (u, v). Two edges are said to be disjoint, if they are not incident with the same vertex. For a subset U of the vertex set V, the subgraph of G induced by U is denoted by $G(U)$ and consists of the subgraph with vertex set U and edge set E_U where E_U contains all edges (u, v) of G with both u and v in U.

For an ordering $v_1, v_2, \ldots, v_p$ of the vertices of G, the adjacency matrix of G is defined to be the $p \times p$ matrix $\operatorname{adj}(G) = M = (m_{ij})$, where $m_{ij} = 1$ if (v_i, v_j) is an edge of G and $m_{ij} = 0$ otherwise. The LU-factorization of $\operatorname{adj}(G)$ is problematic because all of the diagonal entries are zero when the graph has no loops. Furthermore, it may not be desirable to reorder the rows and columns independently, so even if the vertices are reordered the zero entries remain on the diagonal. An approach to LU-factorizations of adjacency matrices which was shown in [2] to have some value, is to partition $\mathrm{adj}(G)$ into 2×2 blocks. One can then consider a block LU-factorization of the partitioned matrix. If the number of vertices is odd, before we partition M into 2×2 blocks, we will add a final isolated vertex for convenience. Denote the result by $\mathrm{adj}_2(G) = A = (a_{ij})$, where A is $n \times n$ with $n = p/2$ and a_{ij} is the ijth submatrix of M.

Next we characterize the graphs G for which $\mathrm{A} = \mathrm{adj}_2(G)$ has an F_3 or EZ-factorization. This will illustrate some of the possibilities, when considering inherited LU-factorizations over rings with zero-divisors. The characterization of graphs for which A has other types of inherited LU-factorizations remains open. First we provide a pair of useful lemmas.

Lemma 6.1. *Let A be $m \times n$ and have an order r EZ-factorization. Let D and E be $m \times m$ and $n \times n$ (respectively) invertible diagonal matrices. Then A^{T}, DA, and AE have order r EZ-factorizations.*

Proof. We use the notation in Theorem 3.6. Let $A = LU$ be an order r EZ-factorization of A. One can easily check that $A^{\mathrm{T}} = (U^{\mathrm{T}} D_1^{-1})(D_1 L^{\mathrm{T}})$ and $AE = L(UE)$ are also order r EZ-factorizations. Writing $D = \begin{bmatrix} D_r & \\ & D_s \end{bmatrix}$, where D_r is $r \times r$, it is also easily seen that $DA = (DLD_r^{-1})(D_r U)$ is an order r EZ-factorization. $\square$

As an application of this lemma consider $D = \mathrm{diag}(a_{11}^{-1}, \ldots, a_{rr}^{-1}, 1, \ldots, 1)$. The first r diagonal entries of matrix DA are 1. Such matrices DA are of our interest, because for these matrices there is no difference between an order r EZ-factorization and an order r G.E. factorization. According to Lemma 6.1 A has an order r EZ-factorization if and only if DA does. In this section we are interested in another application, where A is obtained from a larger matrix M by partitioning M into $k \times k$ submatrices. If $D = \mathrm{diag}(p_1, \ldots, p_m)$ and $E = \mathrm{diag}(q_1, \ldots, q_n)$, where p_i and q_j are $k \times k$ permutation matrices, then DAE is a partitioned form of M with its rows and columns rearranged within the submatrices. By Lemma 6.1 we obtain:

Lemma 6.2. *Suppose that M is a real matrix that can be partitioned into $k \times k$ submatrices, let A denote the matrix whose entries are the submatrices of the partition, and let A' be the matrix whose entries are the submatrices of the partitioned matrix obtained by rearranging the rows and columns of M within the partition. Then A has an order r EZ-factorization over the ring of $k \times k$ real matrices if and only if A' does.*

Define the EZ-graph of order r to be the graph $EZ(r) = (V, E)$, where $V = \{v_1, v_2, \ldots, v_{2r}\}$ and E consists of all unordered pairs (v_i, v_{2j}) with $2j - 1 \leqslant i$. Fig. 1 shows $EZ(4)$.

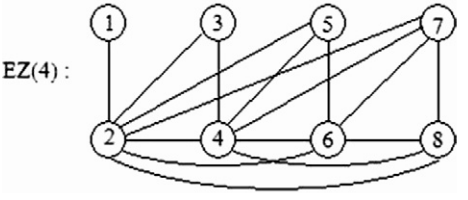

Fig. 1. $EZ(4)$.

At first the adjacency matrix $M = \mathrm{adj}(EZ(r))$ appears to have a modified saw-tooth pattern since $m_{2i-1,k} = 0$ (and $m_{k,2j-1} = 0$) for $k > 2i$ (and for $k > 2j$ respectively). However, the key elements working here are zero divisors and not zeros. For example,

$$A = \mathrm{adj}_2(EZ(4)) = \begin{bmatrix} d & a^{\mathrm{T}} & a^{\mathrm{T}} & a^{\mathrm{T}} \\ a & d & a^{\mathrm{T}} & a^{\mathrm{T}} \\ a & a & d & a^{\mathrm{T}} \\ a & a & a & d \end{bmatrix},$$

where $d = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $a = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. When one checks that A has an EZ-factorization, it is found that the product $ada^{\mathrm{T}} = 0$ plays a key role.

We can now obtain the result:

Theorem 6.3. *Let $A = adj_2(G)$ be the adjacency matrix of a graph G with p vertices, partitioned into 2×2 submatrices. (A final isolated vertex is added if p is odd.) A has an EZ-factorization over the ring of 2×2 real matrices if and only if p is even and for some ordering of its vertices, G is a subgraph of $EZ(r)$ with $r = p/2$ which contains all of the edges $(v_1, v_2), (v_3, v_4), \ldots, (v_{2r-1}, v_{2r})$.*

Proof. We use the notation of Definition 3.5 and Theorem 3.6. Let $d = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ which is the only possibility for an invertible element on the diagonal of A. If A is to have an EZ-factorization, D_1 must be the $r \times r$ matrix $\mathrm{diag}(d, d, \ldots, d)$ with $r = p/2$. This is equivalent to $(v_1, v_2), (v_3, v_4), \ldots, (v_{2r-1}, v_{2r})$ being edges of G. By Theorem 2.2(iii) A has an EZ-factorization if and only if $B_1 D_1^{-1} C_1 = 0$. Since the entries of these matrices are 2×2 matrices of zeros and ones, this condition is equivalent to

$$a_{ik} d a_{kj} = 0, \quad 1 \leqslant k < \min(i, j),\ 1 < i, j \leqslant r. \tag{1}$$

The only solutions to (1) with $i = j$ in the set of 2×2 0, 1-matrices have the form $a_{ik} = \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}$ or $\begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix}$, where $*$ may be 0 or 1. Furthermore, under the condition of (1) the forms of a_{ik} and a_{jk} must agree. That is, once we determine the form of one entry in column k below the diagonal, then (1) requires that all other entries in that column below the diagonal must have the same form. If the second form were to appear, we could use Lemma 6.2 to interchange vertices v_{2k-1} and v_{2k}, which would obtain the first form. The forms of all entries below the diagonal in that column are then swapped. (Entries above the diagonal are determined by symmetry.)

Hence A has an EZ-factorization if and only if $D_1 = \mathrm{diag}(d, d, \ldots, d)$ and

$$a_{ik} = \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix} \quad \text{with } * = 0 \text{ or } 1,\ 1 \leqslant k < i \leqslant r = p/2 \tag{2}$$

for some ordering of the vertices of G. Now (2) only requires that v_{2i-1} and v_{2i} are disjoint from v_{2k-1} for $k < i$ which is equivalent to G being a subgraph of $EZ(r)$. □

While Theorem 6.3 characterizes graphs G with p vertices whose $\mathrm{adj}_2(G)$ have an EZ-factorization of order $p/2$, the more general case of EZ-factorizations of order r with $r < p/2$ remains open. Example 6.4 and Corollary 6.5 suggests some possible directions for such an inquiry.

Example 6.4. Construct a graph G as follows. Start with a subgraph of $EZ(r)$ which contains all of the edges $(v_1, v_2), (v_3, v_4), \dots, (v_{2r-1}, v_{2r})$. To this add the complete bipartite graph with vertex partition $\{v_{2r+1}, v_{2r+3}, \dots, v_{2r+2s-1}\} \bigcup \{v_{2r+2}, v_{2r+4}, \dots, v_{2r+2s}\}$ and then add all of the edges $(v_{2r-1}, v_{2r+2k-1})$ and (v_{2r}, v_{2r+2k}) for $k = 1, 2, \dots, s$. Finally, add the vertices $v_{2r+2s+1}, v_{2r+2s+2}, \dots, v_{2r+2s+t}$ and any number of edges of the form $(v_{2r+2s+i}, v_{2j})$ with $1 \leqslant i \leqslant t$ and $1 \leqslant j < r$.

We can use the above results to show that $\mathrm{adj}_2(G)$ has an order r EZ-factorization. Continuing with the notation of Theorem 3.6, the first part of the construction gives $A_1 = B_1 + D_1 + B_1^{\mathrm{T}}$ which has an EZ-factorization by Theorem 6.3, so that $B_1 D_1 B_1^{\mathrm{T}} = 0$. The second part of the construction provides the leading $s \times s$ submatrix of entries in A_4, namely $a_{ij} = d = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for $r + 1 \leqslant i, j \leqslant r + s$. The second part also gives some of the last column of A_3, namely $a_{rj} = I$ the 2×2 identity matrix for $r + 1 \leqslant j \leqslant r + s$. The remaining entries of A_4 and the last column of A_3 must all be the 2×2 zero matrix by the third part of the construction. The third part of the construction also defines the entries in all other columns of A_3. These all have the form $\begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}$, with $* = 0$ or 1. Now (2) says that the nonzero off-diagonal entries of B_1 also have this form. Hence the entries of $B_1 D_1 A_3^{\mathrm{T}}$ are sums of products of the form $\begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix} d \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}^{\mathrm{T}}$, which are all the 2×2 zero matrix. Hence $B_1 D_1 A_3^{\mathrm{T}} = 0$. Now the ijth entry of $A_3 D_1 A_3^{\mathrm{T}}$ is $a_{r+i,r} d a_{r+j,r}^{\mathrm{T}}$ plus products of the form $\begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix} d \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}^{\mathrm{T}}$. So the ijth entry of $A_3 D_1 A_3^{\mathrm{T}}$ is $IdI^{\mathrm{T}} = d$ for $1 \leqslant i, j \leqslant s$ and 0 otherwise. That is, $A_3 D_1 A_3^{\mathrm{T}} = A_4$. Hence by Theorem 3.6(ii), $\mathrm{adj}_2(G)$ has an order r EZ-factorization.

A *matching* of a graph is a set of disjoint edges. The edges $(v_1, v_2), (v_3, v_4), \dots, (v_{2r-1}, v_{2r})$ provide a matching for the graphs described in Theorem 6.3 and Example 6.4. In Theorem 6.3, but not Example 6.4, the matching is *maximal*. We can obtain the following result.

Corollary 6.5. *Let G be a forest* (*disjoint collection of trees*) *whose maximal matchings contain r edges. Then for some ordering of the vertices*, *$\mathrm{adj}_2(G)$ has an order r EZ-factorization.*

Proof. Order the vertices of G as follows. Choose any vertex of degree 1. Call this vertex v_1, and call the vertex incident to this v_2. Add the edge (v_1, v_2) to a collection M, and remove these two vertices. Continue this process recursively on the remaining part of G until M contains r edges $(v_1, v_2), (v_3, v_4), \dots, (v_{2r-1}, v_{2r})$ which must be a maximal matching of G. Once these are removed the remainder of G consists of isolated vertices which can be numbered $v_{2r+1}, \dots, v_p$ in any order. By the ordering technique there is no edge of the form (v_{2i-1}, v_j) with $1 \leqslant i \leqslant r - 1$, and $2i < j \leqslant 2r$. Hence the subgraph of G induced by the vertices of M is a subgraph of $EZ(r)$ and satisfies the conditions of Theorem 6.3. Any edge of G incident with the vertices $v_{2r+1}, \dots, v_p$ must have the form (v_{2i}, v_{2r+j}) $1 \leqslant i \leqslant r$ so that as in the construction of Example 6.4, $\mathrm{adj}_2(G)$ has an order r EZ-factorization. □

Acknowledgments

The authors greatly appreciate the fact that the referee read the paper very carefully and made many valuable suggestions.

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